the gradient of the velocity component u normal to the wall would be zero, and this conflicts with the shape of the limit curve. The discontinuities at $u/u_0 = 0.25$ and $u/u_0 = 1.00$ would also be appreciably smoothed out by viscosity effects.

If there is a limit curve of this shape, then some explanation would be necessary of how the flow changes to one with a profile showing back-flow at a boundary. Nikuradse found that the profile becomes unsymmetrical, and, for larger angles of divergence, reverse flow takes place along one wall. The unsymmetrical, unseparated flow, called by Nikuradse the "critical separation condition," occurred with angles of divergence between 4.8° and 5.1°. This profile is shown in Fig. 3.

Conclusions

1) The velocity profiles that occur for turbulent flow in two-dimensional, plane-walled diffusers successively approach the profile of a flow that has had the maximum possible momentum loss for a flow symmetrically occupying the full width of the section.

2) It seems probable that this is not accidental.

3) If there is a limit profile for turbulent flow in this case, it would indicate that this mechanism of separation was different from that which occurs in the case of laminar boundary layers, and a distinction between them would have theoretical advantages.

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Sonic Line of Magnetohydrodynamic Nozzle Flow

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Nomenclature

A = Alfvén number
M = Mach number
R = curvature
U = speed of main stream
c = sound velocity
u, v = small-perturbation velocity component
k = specific heat ratio
ϕ = small-perturbation potential function

Subscript

$*$ = critical condition

Introduction

IT has been well known that a nozzle flow of infinitely conducting inviscid gas, with a magnetic field everywhere parallel to the velocity, has four kinds of transitions.[1-3] Among these transitions, the transonic and the transonic-Alfvénic transition are the most interesting regimes for supersonic plasma nozzle design.

Kogan[1] dealt with the transonic transition; however, his solution is incorrect. This paper presents approximate but general rules for the shape of the sonic line for the plane and axisymmetrical nozzle flow of an infinitely conducting inviscid gas with a magnetic field everywhere parallel to the flow velocity.

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The author realized later that Seebass[3] has pointed out Kogan's failure in this regime and has derived the correct solution, however, Seebass does not depict the characteristics of his constant "c".

The procedure used is essentially the same as that of Sauer.[4]

Basic Differential Equations

From the momentum equation, according to the assumptions, Resler[5,6] derived the following differential equations, and later Imai[7] developed them into three-dimensional flow:

$$\frac{(1-M^2)(1-A^2)}{1-A^2-M^2}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{1}$$

$$u=\frac{1-A^2}{1-A^2-M^2}\frac{\partial\phi}{\partial x} \tag{2a}$$

$$v=\partial\phi/\partial y \tag{2b}$$

From the continuity equation, near critical sound velocity, we have

$$-(k+1)\,u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

It is assumed that the velocity components u and v may be expressed as power series in y, with the coefficients of the series depending upon x. Since the flow is symmetrical about the x axis, the series for u must contain only even powers of y, whereas that for v must contain only odd powers. Following the preceding thought, the perturbation potential corresponding to the disturbance velocity u and v is now written in the form

$$\phi=f_0(x)+y^2f_2(x)+y^4f_4(x)+\ldots \tag{4}$$

By differentiation, the velocities are then found to be

$$U_x=U\left(1+\frac{1-A^2}{1-A^2-M^2}\frac{\partial\phi}{\partial x}\right)=a_0+\frac{1}{2!}a_2y^2+\frac{1}{4!}a_4y^4+\ldots \tag{5a}$$

$$U_y=U\frac{\partial\phi}{\partial y}=b_1y+\frac{1}{3!}b^3y^3+\frac{1}{5!}b_5y^5+\ldots \tag{5b}$$

since Eq. (4) is found reasonable.
Substituting Eq. (4) into Eqs. (2), we have

$$u=\frac{1-A^2}{1-M^2-A^2}\frac{\partial\phi}{\partial x}=\frac{1-A^2}{1-A^2-M^2}\times(f_0'+y^2f_2'+y^4f_4'+\ldots) \tag{6a}$$

$$v=\frac{\partial\phi}{\partial y}=2yf_2+4y^3f_4+\ldots \tag{6b}$$

Taking the derivatives $\partial u/\partial x$ and $\partial v/\partial y$ in Eqs. (6), substituting these into Eq. (3), and equating coefficients of like powers of y, we obtain the following relations among the coefficients:

$$f_2=\frac{k+1}{2}\left(\frac{1-A^2}{1-A^2-M^2}\right)^2 f_0'f_0'' \tag{7a}$$

$$f_4=\frac{k+1}{12}\left(\frac{1-A^2}{1-A^2-M^2}\right)^2(f_0'\,f_2''+f_2'f_0'') \tag{7b}$$

Transonic Transition

In this paper, only transonic transition regime is dealt with, i.e., $M = M^* = 1$, $A = A^* \neq 1$. Now f_0 represents, according to Eq. (6a), the velocity distribution on the x axis.

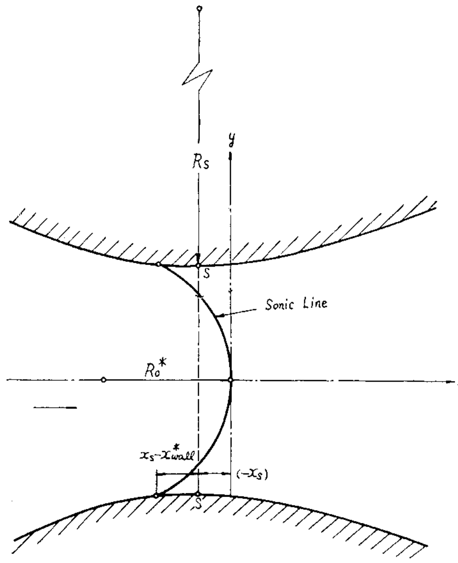

Fig. 1 Sonic line and nomenclature.

For a short distance along the axis, the velocity on the axis may be approximated by a straight line. Restricting our considerations to the neighborhood of the sonic line, and setting $x = 0$ where the sonic line crosses x axis (Fig. 1), we write accordingly

$$f_0' = \frac{A^2}{A^2 - 1} [u]_{y=0} \cong \frac{A^2}{A^2 - 1} \left(\frac{du}{dx}\right)_{y=0} x \tag{8}$$

where $(du/dx)_0$ denotes the velocity gradient on the axis near $x = 0$. Then, from Eqs. (7) and (8), we may rewrite the remaining coefficients as

$$f_2 = \frac{k+1}{2} \left(\frac{du}{dx}\right)_0^2 x \qquad f_4 = \frac{(k+1)^2}{24} \frac{A^2 - 1}{A^2} \left(\frac{du}{dx}\right)_0^3 \tag{9}$$

The velocity distribution [Eqs. (6)] may now be expressed as

$$u = \left(\frac{du}{dx}\right)_0 x + \frac{k+1}{2} \left(\frac{du}{dx}\right)_0^2 \frac{A^2 - 1}{A^2} y^2 + \dots \tag{10a}$$

$$v = (k+1) \left(\frac{du}{dx}\right)_0^2 xy + \frac{(k+1)^2}{6} \left(\frac{du}{dx}\right)_0^3 \frac{A^2 - 1}{A^2} y^3 + \dots \tag{10b}$$

From the condition of sonic line, $(1 + u)^2 + v^2 = 1$, this may be approximated by $u \cong 0$, and we get, from Eq. (10a),

$$x^* = -\frac{k+1}{2} \frac{A^2 - 1}{A^2} \left(\frac{du}{dx}\right)_0 y^{*2} \tag{11}$$

The sonic curve is, therefore, a parabola containing a parameter of Alfvén number A, and its curvature on the x axis may be shown to be given by

$$\frac{1}{R_0^*} = \frac{d^2x^*/dy^{*2}}{\{1 + (dx^*/dy^*)^2\}^{1/2}} = (k+1) \frac{A^2 - 1}{A^2} \left(\frac{du}{dx}\right)_0 \tag{12}$$

The location of the sonic curve to the throat section of the nozzle, signified by point s on the wall, is found from Eq. (10b):

$$-x_s = \frac{k+1}{6} \left(\frac{du}{dx}\right)_0 \frac{A^2 - 1}{A^2} y_s^2 \tag{13}$$

and $(-x_s)$ is the maximum distance of the sonic line downstream of the throat. Setting $y^* \cong y_s$, we may find where the sound speed is first reached on the wall.

Equations (11) and (13) may be combined to yield

$$x_s - x_{\text{wall}}^* = \frac{A^2 - 1}{A^2} \frac{k+1}{3} \left(\frac{du}{dx}\right)_0 y_s^2 = 2(-x_s) \tag{14}$$

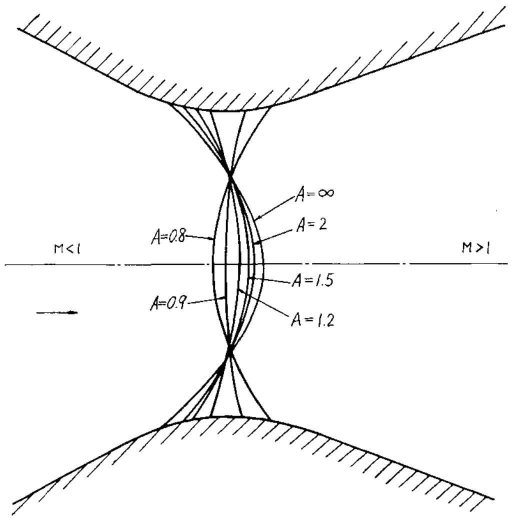

Fig. 2 Shape of sonic line.

The wall curvature near the throat is evaluated as

$$1/R_s \cong \partial v/\partial x$$

then, using Eq. (10b), we have

$$\frac{1}{R_s} = (k+1) \left(\frac{du}{dx}\right)_0^2 y_s = \frac{A^2}{A^2 - 1} \frac{y_s}{R_0^*} \left(\frac{du}{dx}\right)_0 \tag{15}$$

The velocity distribution $(du/dx)_0$ may be defined by y_s and R_s, if we regard the nozzle shape near the throat. Equations (11–15) may be inverted to give the following sonic line parameters in terms of nozzle shape:

$$y_s \left(\frac{du}{dx}\right)_0 = \left\{\frac{y_s}{(k+1)R_s}\right\}^{1/2} \tag{16}$$

$$R_0^* = \frac{A^2}{A^2 - 1} y_s \left\{\frac{R_s}{(k+1)y_s}\right\}^{1/2} \tag{17a}$$

$$x_s - x_{\text{wall}}^* = 2(-x_s) = \frac{A^2 - 1}{A^2} y_s \left\{\frac{(k+1)}{9} \frac{y_s}{R_s}\right\}^{1/2} \tag{17b}$$

Similar considerations for axisymmetric nozzle lead to the following results:

$$R_0^* = \frac{A^2}{A^2 - 1} y_s \left\{\frac{2}{(k+1)} \frac{R_s}{y_s}\right\}^{1/2} \tag{18a}$$

$$x_s - x_{\text{wall}}^* = (-x_s) = \frac{A^2 - 1}{A^2} y_s \left\{\frac{(k+1)}{32} \frac{y_s}{R_s}\right\}^{1/2} \tag{18b}$$

Conclusion

As was seen previously, it has been found that the shape of the sonic line for the plane and axisymmetric nozzle flow of an infinitely conducting inviscid gas with a magnetic field everywhere parallel to the flow velocity is affected by a magnetic field intensity. As we can see from Eqs. (17) and (18), if the Alfvén number is approaching infinity, i.e., a magnetic field intensity approaching zero, the shape of sonic line becomes the same shape of an ordinary nonconducting gas by Sauer,[4] and if the Alfvén number is less than 1, the right-hand side of Eqs. (17) and (18) change their signs. This means that a paraborla convex toward downstream changes into concave toward downstream (see Fig. 2).

These results should be useful for the design of supersonic plasma nozzle exhaust contour.

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Similar Solutions for Three-Dimensional Laminar Compressible Boundary Layers

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Nomenclature

e_1, e_2	= metric coefficients
f, g	= similar functions
h	= enthalpy
H	= stagnation enthalpy, $h + \frac{1}{2}(u^2 + w^2)$
K_1	= curvature parameter, $(2\xi/e_1)(\partial e_1/\partial\zeta)$
K_2	= dilatation parameter, $(2\xi/e_2)(\partial e_2/\partial\xi)$
K^*	= $(\partial/\partial\zeta^*)(\beta^*/rZ)$
M	= Mach number
m	= $[(\gamma - 1)/2]M_e^2$
$\hat{m}$	= $1 + [(\gamma - 1)/2]M_e^2$
N	= $\rho\mu/\rho_0\mu_0$
p	= pressure
P_r	= Prandtl number
r	= body radius
T	= temperature
u, v, w	= velocity components
x, y, z	= curvilinear orthogonal coordinates
α	= $2\xi(dZ/d\zeta)$
β	= pressure gradient parameter, $(2\xi/u_e)(\partial u_e/\partial\xi)\hat{m}$
β^*	= $-(2\xi/\rho_e u_e^2)(\partial p/\partial\zeta)$
ξ, η, ζ	= transformed coordinates
ψ, ϕ	= stream functions
ρ	= density
μ	= dynamic viscosity
θ	= total enthalpy ratio, H/He
ω	= spinning rate

Subscripts

0	= reference state
∞	= freestream condition
e	= outer-edge condition

Superscript

$(\ )'$	= derivative with respect to η

THE three-dimensional boundary layer is characterized by crossflow generation and streamline deviation within the boundary layer. An adequate mathematical description of these phenomena entails the consideration of several additional nonlinear terms in the governing equations which are absent in the corresponding two-dimensional or axisymmetric equations. One approach to this problem is to invoke the concept of the streamline coordinate system for which one of the coordinates is set to be coincident with the local outer-edge streamline projected in the tangent plane. By so doing, the perturbation technique in treating the crossflow effect can be justified for a wide variety of practical applications, and because of the presence of only one external flow component, the construction of a single similarity parameter can be achieved without introducing any restrictive assumptions. The purpose of this note is to discuss a certain class of similar solutions for three-dimensional laminar compressible boundary-layer flows on the basis of the streamline coordinate system and by employing a set of simple transformation variables. Neglected herein are the real gas effects associated with hypersonic flight, flow separation, and laminar-to-turbulent transition phenomena, and the analytical procedures for the determination of surface streamlines in accordance with the theory of differential geometry.

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Similar to the consideration by Beckwith,[1] we select a three-dimensional orthogonal streamline coordinate system (x,y,z) with corresponding velocity components u tangent to the external streamline, v normal to the body surface, and w, the crossflow velocity component, normal to u in the tangent plane. The length elements are

$$ds = e_1(x, z)dx \qquad dy = dy \qquad dn = e_2(x, z)dz \quad (1)$$

and at the outer edge of the boundary layer, $u = u_e$ and $w = w_e = 0$. The governing equations for a three-dimensional laminar compressible boundary layer may then be written as:

Continuity

$$\frac{1}{e_1}\frac{\partial}{\partial x}(\rho u e_2) + \frac{\partial}{\partial y}(\rho v e_2) + \frac{1}{e_1}\frac{\partial}{\partial z}(\rho w e_1) = 0 \quad (2)$$

x Momentum

$$\frac{\rho u}{e_1}\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \frac{\rho w}{e_2}\frac{\partial u}{\partial z} + \frac{\rho u w}{e_1 e_2}\frac{\partial e_1}{\partial z} - \frac{\rho w^2}{e_1 e_2}\frac{\partial e_2}{\partial x} = -\frac{1}{e_1}\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (3)$$

y Momentum

$$\partial p/\partial y = 0 \quad (4)$$

z Momentum

$$\frac{\rho u}{e_1}\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \frac{\rho w}{e_2}\frac{\partial w}{\partial z} + \frac{\rho u w}{e_1 e_2}\frac{\partial e_2}{\partial x} - \frac{\rho u^2}{e_1 e_2}\frac{\partial e_1}{\partial z} = -\frac{1}{e_2}\frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \quad (5)$$

Energy

$$\frac{\rho u}{e_1}\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} + \frac{\rho w}{e_2}\frac{\partial H}{\partial z} = \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial H}{\partial y} + \frac{1 - P_r}{P_r}\frac{\partial h}{\partial y}\right)\right] \quad (6)$$

From the continuity equation, we define two stream functions ψ and ϕ as follows:

$$\rho u e_2 = \frac{\partial\psi}{\partial y} \qquad \rho v e_2 = \left(-\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial z}\right)\frac{1}{e_1} \qquad \rho w e_1 = \frac{\partial\phi}{\partial y} \quad (7)$$

We now introduce the transformation variables

$$\xi = \int_0^x \rho_0\mu_0 e_1 dx = \int_0^s \rho_0\mu_0 ds$$
$$\eta = \left(\frac{u_e}{2\xi}\right)^{1/2}\int_0^y \rho dy \quad (8)$$
$$\zeta = \int_0^z \rho_0\mu_0 e_2 dz = \int_0^n \rho_0\mu_0 dn$$